

# Factorization of the characteristic function of a Jacobi matrix

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## Abstract

In a recent paper a class of infinite Jacobi matrices with discrete character of spectra has been introduced. With each Jacobi matrix from this class an analytic function is associated, called the characteristic function, whose zero set coincides with the point spectrum of the corresponding Jacobi operator. Here it is shown that the characteristic function admits Hadamard's factorization in two possible ways – either in the spectral parameter or in an auxiliary parameter which may be called the coupling constant. As an intermediate result, an explicit expression for the power series expansion of the logarithm of the characteristic function is obtained.

*Keywords:* infinite Jacobi matrix, characteristic function, Hadamard's factorization

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## 1 Introduction

In [12] we have introduced a class of infinite Jacobi matrices characterized by a simple convergence condition. Each Jacobi matrix from this class unambiguously determines a closed operator on  $\ell^2(\mathbb{N})$  having a discrete spectrum. Moreover, with such a matrix one associates a complex function, called the characteristic function, which is analytic on the complex plane with the closure of the range of the diagonal sequence being excluded, and meromorphic on the complex plane with the set of accumulation points of the diagonal sequence being excluded. It turns out that the zero set of the characteristic function actually coincides with the point spectrum of the corresponding Jacobi operator on the domain of definition (with some subtleties when handling the poles; see Theorem 1 below).

The aim of the current paper is to show that the characteristic function admits Hadamard's factorization in two possible ways. First, assuming that the Jacobi matrix is real and the corresponding operator self-adjoint, we derive a factorization in the spectral parameter. Further, for symmetric complex Jacobi matrices we assume the off-diagonal elements to depend linearly on an auxiliary parameter which we call, following physical terminology, the coupling constant. The second factorization formula then concerns this parameter.

Many formulas throughout the paper are expressed in terms of a function, called  $\mathfrak{F}$ , which is defined on a suitable subset of the linear space of all complex sequences; see [11] for its original definition. This function was also heavily employed in [12]. So we start from recalling its definition and basic properties. Apart of the announced Hadamard factorization we derive, as an intermediate step, a formula for  $\log \mathfrak{F}(x)$ .

Define  $\mathfrak{F} : D \rightarrow \mathbb{C}$ ,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \cdots x_{k_m} x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C}; \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}. \quad (1)$$

For a finite number of complex variables we identify  $\mathfrak{F}(x_1, x_2, \dots, x_n)$  with  $\mathfrak{F}(x)$  where  $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$ . By convention, let  $\mathfrak{F}(\emptyset) = 1$  where  $\emptyset$  is the empty sequence.

Notice that  $\ell^2(\mathbb{N}) \subset D$ . For  $x \in D$ , one has the estimates

$$|\mathfrak{F}(x)| \leq \exp \left( \sum_{k=1}^{\infty} |x_k x_{k+1}| \right), \quad |\mathfrak{F}(x) - 1| \leq \exp \left( \sum_{k=1}^{\infty} |x_k x_{k+1}| \right) - 1, \quad (2)$$

and it is true that

$$\mathfrak{F}(x) = \lim_{n \rightarrow \infty} \mathfrak{F}(x_1, x_2, \dots, x_n). \quad (3)$$

Let us also point out a simple invariance property. For  $x \in D$  and  $s \in \mathbb{C}$ ,  $s \neq 0$ , it is true that  $y \in D$  and

$$\mathfrak{F}(x) = \mathfrak{F}(y), \text{ where } y_{2k-1} = s x_{2k-1}, \ y_{2k} = x_{2k}/s, \ k \in \mathbb{N}. \quad (4)$$

We shall deal with symmetric Jacobi matrices

$$J = \begin{bmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \quad (5)$$

where  $\lambda = \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$  and  $w = \{w_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ . Let us put

$$\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}, \quad \gamma_{2k} = w_1 \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}, \quad k = 1, 2, 3, \dots \quad (6)$$

Then  $\gamma_k \gamma_{k+1} = w_k$ .

For  $n \in \mathbb{N}$ , let  $J_n$  be the  $n \times n$  Jacobi matrix:  $(J_n)_{j,k} = J_{j,k}$  for  $1 \leq j, k \leq n$ , and  $I_n$  be the  $n \times n$  unit matrix. Then the formula

$$\det(J_n - zI_n) = \left( \prod_{k=1}^n (\lambda_k - z) \right) \mathfrak{F} \left( \frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \dots, \frac{\gamma_n^2}{\lambda_n - z} \right). \quad (7)$$

holds true for all  $z \in \mathbb{C}$  (after obvious cancellations, the RHS is well defined even for  $z = \lambda_k$ ; here and throughout RHS means “right-hand side”, and similarly for LHS).

Let us denote

$$\mathbb{C}_0^\lambda := \mathbb{C} \setminus \overline{\{\lambda_n; n \in \mathbb{N}\}}.$$

Moreover,  $\text{der}(\lambda)$  designates the set of all accumulation points of the sequence  $\lambda$ . The following theorem is a compilation of several results from [12, Subsec. 3.3].

**Theorem 1.** *Let a Jacobi matrix  $J$  be real and suppose that*

$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \right| < \infty \quad (8)$$

for at least one  $z \in \mathbb{C}_0^\lambda$ . Then

- (i)  $J$  represents a unique self-adjoint operator on  $\ell^2(\mathbb{N})$ ,
- (ii)  $\text{spec}(J) \cap (\mathbb{C} \setminus \text{der}(\lambda))$  consists of simple real eigenvalues with no accumulation points in  $\mathbb{C} \setminus \text{der}(\lambda)$ ,
- (iii) the series (8) converges locally uniformly on  $\mathbb{C}_0^\lambda$  and

$$F_J(z) := \mathfrak{F} \left( \left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \right) \quad (9)$$

is a well defined analytic function on  $\mathbb{C}_0^\lambda$ ,

- (iv)  $F_J(z)$  is meromorphic on  $\mathbb{C} \setminus \text{der}(\lambda)$ , the order of a pole at  $z \in \mathbb{C} \setminus \text{der}(\lambda)$  is less than or equal to the number  $r(z)$  of occurrences of  $z$  in the sequence  $\lambda$ ,
- (v)  $z \in \mathbb{C} \setminus \text{der}(\lambda)$  belongs to  $\text{spec}(J)$  if and only if

$$\lim_{u \rightarrow z} (z - u)^{r(z)} F_J(u) = 0$$

and, in particular,  $\text{spec}(J) \cap \mathbb{C}_0^\lambda = \text{spec}_p(J) \cap \mathbb{C}_0^\lambda = F_J^{-1}(\{0\})$ .

We will mostly focus on real Jacobi matrices, except Section 4. For our purposes the following particular case, a direct consequence of a more general result derived in [12, Subsec. 3.3], will be sufficient.

**Theorem 2.** *Let  $J$  be a complex Jacobi matrix of the form (5) obeying  $\lambda_n = 0$ ,  $\forall n$ , and  $\{w_n\} \in \ell^2(\mathbb{N})$ . Then  $J$  represents a Hilbert-Schmidt operator,  $F_J(z)$  is analytic on  $\mathbb{C} \setminus \{0\}$  and*

$$\text{spec}(J) \setminus \{0\} = \text{spec}_p(J) \setminus \{0\} = F_J^{-1}(\{0\}).$$

## 2 The logarithm of $\mathfrak{F}(x)$

$\mathfrak{F}(x_1, \dots, x_n)$  is a polynomial function in  $n$  complex variables, with  $\mathfrak{F}(0) = 1$ , and so  $\log \mathfrak{F}(x_1, \dots, x_n)$  is a well defined analytic function in some neighborhood of the origin. The goal of the current section is to derive an explicit formula for the coefficients of the corresponding power series.

For a multiindex  $m \in \mathbb{N}^\ell$  denote by  $|m| = \sum_{j=1}^\ell m_j$  its order and by  $d(m) = \ell$  its length. For  $N \in \mathbb{N}$  define

$$\mathcal{M}(N) = \left\{ m \in \bigcup_{\ell=1}^N \mathbb{N}^\ell; |m| = N \right\}. \quad (10)$$

Obviously,  $\cup_{\ell=1}^\infty \mathbb{N}^\ell = \cup_{N=1}^\infty \mathcal{M}(N)$ . One has  $\mathcal{M}(1) = \{(1)\}$  and

$$\begin{aligned} \mathcal{M}(N) = & \left\{ (1, m_1, m_2, \dots, m_{d(m)}) ; m \in \mathcal{M}(N-1) \right\} \\ & \cup \left\{ (m_1 + 1, m_2, \dots, m_{d(m)}) ; m \in \mathcal{M}(N-1) \right\}. \end{aligned}$$

Hence  $|\mathcal{M}(N)| = 2^{N-1}$  ( $|\cdot|$  standing for the number of elements). Furthermore, for an multiindex  $m \in \mathbb{N}^\ell$  put

$$\beta(m) := \prod_{j=1}^{\ell-1} \binom{m_j + m_{j+1} - 1}{m_{j+1}}, \quad \alpha(m) := \frac{\beta(m)}{m_1}. \quad (11)$$

**Proposition 3.** *In the ring of formal power series in the variables  $t_1, \dots, t_n$ , one has*

$$\log \mathfrak{F}(t_1, \dots, t_n) = - \sum_{\ell=1}^{n-1} \sum_{m \in \mathbb{N}^\ell} \alpha(m) \sum_{k=1}^{n-\ell} \prod_{j=1}^{\ell} (t_{k+j-1} t_{k+j})^{m_j}. \quad (12)$$

For a complex sequence  $x = \{x_k\}_{k=1}^\infty$  such that  $\sum_{k=1}^\infty |x_k x_{k+1}| < \log 2$  one has

$$\log \mathfrak{F}(x) = - \sum_{\ell=1}^\infty \sum_{m \in \mathbb{N}^\ell} \alpha(m) \sum_{k=1}^\infty \prod_{j=1}^{\ell} (x_{k+j-1} x_{k+j})^{m_j}.$$

The proof of Proposition 3 is based on some combinatorial notions among them that of Dyck path is quite substantial. For  $n \in \mathbb{N}$ ,  $n \geq 2$ , we may regard the set

$$\Lambda_n = \{1, 2, \dots, n\}$$

as a finite one-dimensional lattice. We shall say that a mapping

$$\pi : \{0, 1, 2, \dots, 2N\} \rightarrow \Lambda_n$$

is a loop of length  $2N$  in  $\Lambda_n$ ,  $N \in \mathbb{N}$ , if  $\pi(0) = \pi(2N)$  and  $|\pi(j+1) - \pi(j)| = 1$  for  $1 \leq j \leq 2N$ . The vertex  $\pi(0)$  is called the base point of a loop. The loops in  $\Lambda_n$  with

the base point  $\pi(0) = 1$  are commonly known as Dyck paths of height not exceeding  $n - 1$ . Indeed, if  $\pi$  is such a loop then its graph shifted by 1,

$$\{(j, \pi(j) - 1); j = 0, 1, \dots, 2N\},$$

represents a lattice path in the first quadrant leading from  $(0, 0)$  to  $(2N, 0)$  whose all steps are solely  $(1, 1)$  and  $(1, -1)$ . Such a path is called a Dyck path.

For  $m \in \mathbb{N}^\ell$  denote by  $\Omega(m)$  the set of all loops of length  $2|m|$  in  $\Lambda_{\ell+1}$  which encounter each edge  $(j, j+1)$  exactly  $2m_j$  times,  $1 \leq j \leq \ell$  (counting both directions). Let  $\Omega_1(m)$  designate the subset of  $\Omega(m)$  formed by those loops which are based at the vertex 1. In other words,  $\Omega_1(m)$  is the set of Dyck paths with the prescribed numbers  $2m_j$  counting the steps at each level  $j = 1, 2, \dots, \ell$ . One can call  $m$  the specification of a Dyck path. If  $\pi \in \Omega_1(m)$  then the sequence  $(\pi(0), \pi(1), \dots, \pi(2N - 1))$ , with  $N = |m|$ , contains the vertex 1 exactly  $m_1$  times, the vertices  $j$ ,  $2 \leq j \leq \ell$ , are contained  $(m_{j-1} + m_j)$  times in the sequence, and the number of occurrences of the vertex  $\ell + 1$  equals  $m_\ell$ .

*Remark 4.* It can be deduced from Theorem 3B in [4] that  $|\Omega_1(m)| = \beta(m)$ . Let us recall the well known fact that there exists a bijection between the set of Dyck paths of length  $2N$  and the set of rooted plane trees with  $N$  edges (one can consult, for instance, §§ I.5 and I.6 in [5]). A rooted plane tree is said to have the specification  $m \in \mathbb{N}^\ell$  if it has  $|m|$  edges and the number of its vertices of height  $j$  equals  $m_j$ ,  $j = 1, 2, \dots, \ell$ . Using the mentioned bijection one finds that  $\beta(m)$  also equals the number of rooted plane trees with the specification  $m$  [4, 9]. More recently, this result has been rediscovered and described in [1]. For the reader's convenience we nevertheless include this identity in the following lemma along with a short proof. The other identity in the lemma providing a combinatorial interpretation of the number  $\alpha(m)$  seems to be, to the authors' best knowledge, new.

**Lemma 5.** *For every  $\ell \in \mathbb{N}$  and  $m \in \mathbb{N}^\ell$ ,  $|\Omega_1(m)| = \beta(m)$  and  $|\Omega(m)| = 2|m|\alpha(m)$ .*

*Proof.* To show the first equality one can proceed by induction in  $\ell$ . For  $\ell = 1$  and any  $m \in \mathbb{N}$  one clearly has  $|\Omega_1(m)| = 1$ . Suppose now that  $\ell \geq 2$  and fix  $m \in \mathbb{N}^\ell$ . Denote  $m' = (m_2, \dots, m_\ell) \in \mathbb{N}^{\ell-1}$ . For any  $\pi' \in \Omega_1(m')$  put

$$\tilde{\pi} = (1, \pi'(0) + 1, \pi'(1) + 1, \dots, \pi'(2N') + 1, 1)$$

where  $N' = |m'| = |m| - m_1$ . The vertex 2 occurs in  $\tilde{\pi}$  exactly  $(m_2 + 1)$  times. After any such occurrence of 2 one may insert none or several copies of the two-letter chain  $(1, 2)$ . Do it so while requiring that the total number of inserted couples equals  $m_1 - 1$ . This way one generates all Dyck paths from  $\Omega_1(m)$ , and each exactly once. This implies the recurrence rule

$$|\Omega_1(m_1, m_2, \dots, m_\ell)| = \binom{m_1 - 1 + m_2}{m_2} |\Omega_1(m_2, \dots, m_\ell)|,$$

thus proving that  $|\Omega_1(m)| = \beta(m)$ .

Let us proceed to the second equality. Put  $N = |m|$ . Consider the cyclic group  $G = \langle g \rangle$ ,  $g^{2N} = 1$ .  $G$  acts on  $\Omega(m)$  according to the rule

$$g \cdot \pi = (\pi(1), \pi(2), \dots, \pi(2N), \pi(0)), \quad \forall \pi \in \Omega(m).$$

Clearly,  $G \cdot \Omega_1(m) = \Omega(m)$ . Let us write  $\Omega(m)$  as a disjoint union of orbits,

$$\Omega(m) = \bigcup_{s=1}^M \mathcal{O}_s.$$

For each orbit choose  $\pi_s \in \mathcal{O}_s \cap \Omega_1(m)$ . Let  $H_s \subset G$  be the stabilizer of  $\pi_s$ . Then

$$|\Omega(m)| = \sum_{s=1}^M \frac{2N}{|H_s|}.$$

Denote further by  $G_s^1$  the subset of  $G$  formed by those elements  $a$  obeying  $a \cdot \pi_s \in \Omega_1(m)$  (i.e. the vertex 1 is still the base point). Then  $|G_s^1| = m_1$  and  $\mathcal{O}_s \cap \Omega_1(m) = G_s^1 \cdot \pi_s$ . Moreover,  $G_s^1 \cdot H_s = G_s^1$ , i.e.  $H_s$  acts freely from the right on  $G_s^1$ , with orbits of this action being in one-to-one correspondence with elements of  $\mathcal{O}_s \cap \Omega_1(m)$ . Hence  $|\mathcal{O}_s \cap \Omega_1(m)| = |G_s^1|/|H_s|$  and

$$|\Omega_1(m)| = \sum_{s=1}^M |\mathcal{O}_s \cap \Omega_1(m)| = \sum_{s=1}^M \frac{m_1}{|H_s|}.$$

This shows that  $|\Omega(m)| = (2N/m_1)|\Omega_1(m)|$ . In view of the first equality of the proposition and (11), the proof is complete.  $\square$

**Lemma 6.** For  $N \in \mathbb{N}$ ,

$$\sum_{m \in \mathcal{M}(N)} \alpha(m) = \frac{1}{2N} \binom{2N}{N}.$$

*Proof.* According to Lemma 5, the sum

$$2N \sum_{m \in \mathcal{M}(N)} \alpha(m) = \sum_{m \in \mathcal{M}(N)} |\Omega(m)|$$

equals the number of equivalence classes of loops of length  $2N$  in the one-dimensional lattice  $\mathbb{Z}$  assuming that loops differing by translations are identified. These classes are generated by making  $2N$  choices, in all possible ways, each time choosing either the sign plus or minus (moving to the right or to the left on the lattice) while the total number of occurrences of each sign being equal to  $N$ .  $\square$

*Remark 7.* The sum  $\sum_{m \in \mathcal{M}(N)} \beta(m)$  can readily be evaluated, too, since this is nothing but the total number of Dyck paths of length  $2N$ . As is well known, this number equals the Catalan number

$$C_N := \frac{1}{N+1} \binom{2N}{N}$$

(see, for instance [3]).

For  $m \in \mathbb{N}^\ell$  let

$$\binom{|m|}{m} := \frac{|m|!}{m_1! m_2! \cdots m_\ell!}.$$

**Lemma 8.** *For every  $\ell \in \mathbb{N}$  and  $m \in \mathbb{N}^\ell$ ,*

$$\alpha(m) \leq \frac{1}{|m|} \binom{|m|}{m},$$

*and equality holds if and only if  $\ell = 1$  or  $2$ .*

*Proof.* Put  $\gamma(m) = \alpha(m) / \binom{|m|}{m}$ . To show that  $\gamma(m) \leq 1/|m|$  one can proceed by induction in  $\ell$ . It is immediate to check the equality to be true for  $\ell = 1$  and  $2$ . For  $\ell \geq 3$  and  $m_1 > 1$  one readily verifies that

$$\gamma(m_1, m_2, m_3, \dots, m_\ell) < \gamma(m_1 - 1, m_2 + 1, m_3, \dots, m_\ell).$$

Furthermore, if  $\ell \geq 3$ ,  $m_1 = 1$  and the inequality is known to be valid for  $\ell - 1$ , one has

$$\gamma(m_1, m_2, m_3, \dots, m_\ell) = \frac{m_2 \gamma(m_2, m_3, \dots, m_\ell)}{1 + m_2 + m_3 + \cdots + m_\ell} < \frac{1}{|m|}.$$

The lemma follows.  $\square$

*Proof of Proposition 3.* The coefficients of the power series expansion at the origin of the function  $\log \mathfrak{F}(t_1, \dots, t_n)$  can be calculated in the ring of formal power series. As shown in [12], one has

$$\mathfrak{F}(t_1, \dots, t_n) = \det(I + T)$$

where

$$T = \begin{bmatrix} 0 & t_1 & & & & \\ t_2 & 0 & t_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & t_{n-1} & 0 & t_{n-1} \\ & & & & t_n & 0 \end{bmatrix}. \quad (13)$$

Since  $\det \exp(A) = \exp(\text{Tr } A)$  and so  $\log \det(I + T) = \text{Tr } \log(I + T)$ , and noticing that  $\text{Tr } T^{2k+1} = 0$ , one gets

$$\log \mathfrak{F}(t_1, \dots, t_n) = \text{Tr } \log(I + T) = - \sum_{N=1}^{\infty} \frac{1}{2N} \text{Tr } T^{2N}.$$

From (13) one deduces that

$$\text{Tr } T^{2N} = \sum_{\pi \in \mathcal{L}(N)} \prod_{j=0}^{2N-1} t_{\pi(j)} \quad (14)$$

where  $\mathcal{L}(N)$  stands for the set of all loops of length  $2N$  in  $\Lambda_n$ . Let

$$k = \min\{\pi(j); 1 \leq j \leq 2N\}$$

and put  $\tilde{\pi}(j) = \pi(j) - k + 1$  for  $0 \leq j \leq 2N$ . Then  $\tilde{\pi} \in \Omega(m)$  for certain (unambiguous) multiindex  $m \in \mathcal{M}(N)$  of length  $d(m) \leq n - k$ . Conversely, given  $m \in \mathcal{M}(N)$  of length  $d(m) \leq n - 1$  and  $k$ ,  $1 \leq k \leq n - d(m)$ , one defines  $\pi \in \mathcal{L}(N)$  by  $\pi(j) = k + \tilde{\pi}(j) - 1$ ,  $0 \leq j \leq 2N$ . Hence the RHS of (14) equals

$$\sum_{\substack{m \in \mathcal{M}(N) \\ d(m) < n}} \sum_{k=1}^{n-d(m)} |\Omega(m)| \prod_{j=1}^{d(m)} (t_{k+j-1} t_{k+j})^{m_j}.$$

To verify (12) it suffices to apply Lemma 5.

Suppose now  $x$  is a complex sequence. If  $\sum_k |x_k x_{k+1}| < \log 2$  one has, by (2),  $|\mathfrak{F}(x) - 1| < 1$  and so  $\log \mathfrak{F}(x)$  is well defined. Moreover, according to (3),

$$\log \mathfrak{F}(x) = \lim_{n \rightarrow \infty} \log \mathfrak{F}(x_1, \dots, x_n).$$

If  $\sum_k |x_k x_{k+1}| < 1$  then the RHS of (12) admits the limit procedure, too, as demonstrated by the simple estimate (replacing  $t_j$ s by  $x_j$ s)

$$\begin{aligned} |\text{the RHS of (12)}| &\leq \sum_{N=1}^{\infty} \left( \max_{m \in \mathcal{M}(N)} \frac{\alpha(m)}{\binom{N}{m}} \right) \sum_{m \in \mathcal{M}(N)} \binom{N}{m} \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} |x_{k+j-1} x_{k+j}|^{m_j} \\ &\leq \sum_{N=1}^{\infty} \frac{1}{N} \left( \sum_{k=1}^{\infty} |x_k x_{k+1}| \right)^N = -\log \left( 1 - \sum_{k=1}^{\infty} |x_k x_{k+1}| \right). \end{aligned}$$

Here we have used Lemma 8. □

### 3 Factorization in the spectral parameter

In this section, we introduce a regularized characteristic function of a Jacobi matrix and show that it can be expressed as a Hadamard infinite product.

Let  $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ ,  $\{w_n\}_{n=1}^{\infty}$  be real sequences such that  $\lim_{n \rightarrow \infty} \lambda_n = +\infty$  and  $w_n \neq 0$ ,  $\forall n$ . In addition, without loss of generality,  $\{\lambda_n\}_{n=1}^{\infty}$  is assumed to be positive. Moreover, suppose that

$$\sum_{n=1}^{\infty} \frac{w_n^2}{\lambda_n \lambda_{n+1}} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty. \quad (15)$$

Under these assumptions, by Theorem 1,  $J$  defined in (5) may be regarded as a self-adjoint operator on  $\ell^2(\mathbb{N})$ . Moreover,  $\text{der}(\lambda)$  is clearly empty and the characteristic



function  $F_J(z)$  is meromorphic on  $\mathbb{C}$  with possible poles lying in the range of  $\lambda$ . To remove the poles let us define the function

$$\Phi_\lambda(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}.$$

Since  $\sum_n \lambda_n^{-2} < \infty$ ,  $\Phi_\lambda$  is a well defined entire function. Moreover,  $\Phi_\lambda$  has zeros at the points  $z = \lambda_n$ , with multiplicity being equal to the number of repetitions of  $\lambda_n$  in the sequence  $\lambda$ , and no zeros otherwise.

Finally we define (see (9))

$$H_J(z) := \Phi_\lambda(z) F_J(z),$$

and call  $H_J(z)$  the regularized characteristic function of the Jacobi operator  $J$ . Note that for  $\varepsilon \geq 0$ ,  $F_{J+\varepsilon I}(z) = F_J(z - \varepsilon)$  and so

$$H_{J+\varepsilon I}(z) = H_J(z - \varepsilon) \Phi_\lambda(-\varepsilon)^{-1} \exp\left(-z \sum_{n=1}^{\infty} \frac{\varepsilon}{\lambda_n(\lambda_n + \varepsilon)}\right). \quad (16)$$

According to Theorem 1, the spectrum of  $J$  is discrete, simple and real. Moreover,

$$\text{spec}(J) = \text{spec}_p(J) = H_J^{-1}(\{0\}).$$

As is well known, the determinant of an operator  $I + A$  on a Hilbert space can be defined provided  $A$  belongs to the trace class. The definition, in a modified form, can be extended to other Schatten classes  $\mathcal{S}_p$  as well, in particular to Hilbert-Schmidt operators; see [10] for a detailed survey of the theory. Let us denote, as usual, the trace class and the Hilbert-Schmidt class by  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , respectively. If  $A \in \mathcal{S}_2$  then

$$(I + A) \exp(-A) - I \in \mathcal{S}_1,$$

and one defines

$$\det_2(I + A) := \det((I + A) \exp(-A)).$$

We shall need the following formulas [10, Chp. 9]. For  $A, B \in \mathcal{S}_2$  one has

$$\det_2(I + A + B + AB) = \det_2(I + A) \det_2(I + B) \exp(-\text{Tr}(AB)). \quad (17)$$

A factorization formula holds for  $A \in \mathcal{S}_2$  and  $z \in \mathbb{C}$ ,

$$\det_2(I + zA) = \prod_{n=1}^{N(A)} (1 + z\mu_n(A)) \exp(-z\mu_n(A)), \quad (18)$$

where  $\mu_n(A)$  are all (nonzero) eigenvalues of  $A$  counted up to their algebraic multiplicity (see Theorem 9.2 in [10] and also Theorem 1.1 ibidem introducing the algebraic multiplicity of a nonzero eigenvalue of a compact operator). In particular,  $I + zA$  is

invertible if and only if  $\det_2(I + zA) \neq 0$ . Moreover, the Plemejl-Smithies formula tells us that for  $A \in \mathcal{J}_2$ ,

$$\det_2(I + zA) = \sum_{m=0}^{\infty} a_m(A) \frac{z^m}{m!}, \quad (19)$$

where

$$a_m(A) = \det \begin{bmatrix} 0 & m-1 & 0 & \dots & 0 & 0 \\ \text{Tr } A^2 & 0 & m-2 & \dots & 0 & 0 \\ \text{Tr } A^3 & \text{Tr } A^2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \text{Tr } A^{m-1} & \text{Tr } A^{m-2} & \text{Tr } A^{m-3} & \dots & 0 & 1 \\ \text{Tr } A^m & \text{Tr } A^{m-1} & \text{Tr } A^{m-2} & \dots & \text{Tr } A^2 & 0 \end{bmatrix} \quad (20)$$

for  $m \geq 1$ , and  $a_0(A) = 1$  [10, Thm. 5.4]. Finally, there exists a constant  $C_2$  such that for all  $A, B \in \mathcal{J}_2$ ,

$$|\det_2(I + A) - \det_2(I + B)| \leq \|A - B\|_2 \exp(C_2(\|A\|_2 + \|B\|_2 + 1)^2), \quad (21)$$

where  $\|\cdot\|_2$  stands for the Hilbert-Schmidt norm.

We write the Jacobi matrix in the form

$$J = L + W + W^*$$

where  $L$  is a diagonal matrix while  $W$  is lower triangular. By assumption (15), the operators  $L^{-1}$  and

$$K := L^{-1/2}(W + W^*)L^{-1/2} \quad (22)$$

are Hilbert-Schmidt. Hence for every  $z \in \mathbb{C}$ , the operator  $L^{-1/2}(W + W^* - z)L^{-1/2}$  belongs to the Hilbert-Schmidt class.

**Lemma 9.** *For every  $z \in \mathbb{C}$ ,*

$$H_J(z) = \det_2(I + L^{-1/2}(W + W^* - z)L^{-1/2}).$$

*In particular,*

$$H_J(0) = F_J(0) = \det_2(I + K).$$

*Proof.* We first verify the formula for the truncated finite rank operator  $J_N = P_N J P_N$ , where  $P_N$  is the orthogonal projection onto the subspace spanned by the first  $N$  vectors of the canonical basis in  $\ell^2(\mathbb{N})$ . Using formula (7) one derives

$$\begin{aligned} & \det[(I + P_N L^{-1/2}(W + W^* - z)L^{-1/2}P_N) \exp(-P_N L^{-1/2}(W + W^* - z)L^{-1/2}P_N)] \\ &= \det(P_N L^{-1}P_N) \det(J_N - zI_N) \exp(z \text{Tr}(P_N L^{-1}P_N)) \\ &= \left( \prod_{n=1}^N \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n} \right) \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^N\right). \end{aligned}$$

Sending  $N$  to infinity it is clear, by (3) and (15), that the RHS tends to  $H_J(z)$ . Moreover, one knows that  $\det_2(I + A)$  is continuous in  $A$  in the Hilbert-Schmidt norm, as it follows from (21). Thus to complete the proof it suffices to notice that  $A \in \mathcal{J}_2$  implies  $\|P_N A P_N - A\|_2 \rightarrow 0$  as  $N \rightarrow \infty$ .  $\square$

We intend to apply the Hadamard factorization theorem to  $H_J(z)$ ; see, for example, [2, Thm. XI.3.4]. For simplicity we assume that  $F_J(0) \neq 0$  and so  $J$  is invertible. Otherwise one could replace  $J$  by  $J + \varepsilon I$  for some  $\varepsilon > 0$  and make use of (16).

As already mentioned, the operator  $K$  defined in (22) is Hilbert-Schmidt. At the same time, this is a Jacobi matrix operator with zero diagonal admitting application of Theorem 1. One readily finds that

$$F_K(z) = \mathfrak{F}\left(\left\{-\frac{\gamma_n^2}{z\lambda_n}\right\}_{n=1}^{\infty}\right).$$

Hence  $F_K(-1) = F_J(0)$ , and  $J$  is invertible if and only if the same is true for  $(I + K)$ . In that case, again by Theorem 1, 0 belongs to the resolvent set of  $J$ , and

$$J^{-1} = L^{-1/2}(I + K)^{-1}L^{-1/2}. \quad (23)$$

**Lemma 10.** *If  $J$  is invertible then  $J^{-1}$  is a Hilbert-Schmidt operator and*

$$\det_2(I - z(I + K)^{-1}L^{-1}) = \det_2(I - zJ^{-1}) \quad (24)$$

for all  $z \in \mathbb{C}$ .

*Proof.* By assumption (15),  $L^{-1/2}$  belongs to the Schatten class  $\mathcal{S}_4$ . Since the Schatten classes are norm ideals and fulfill  $\mathcal{S}_p\mathcal{S}_q \subset \mathcal{S}_r$  whenever  $r^{-1} = p^{-1} + q^{-1}$  [10, Thm. 2.8], one deduces from (23) that  $J^{-1} \in \mathcal{S}_2$ .

Furthermore, one knows that  $\text{Tr}(AB) = \text{Tr}(BA)$  provided  $A \in \mathcal{S}_p$ ,  $B \in \mathcal{S}_q$  and  $p^{-1} + q^{-1} = 1$  [10, Cor. 3.8]. Hence

$$\text{Tr}((I + K)^{-1}L^{-1})^k = \text{Tr}(L^{-1/2}(I + K)^{-1}L^{-1/2})^k = \text{Tr}(J^{-k}), \quad \forall k \in \mathbb{N}, k \geq 2.$$

It follows that coefficients  $a_m$  defined in (20) fulfill

$$a_m((I + K)^{-1}L^{-1}) = a_m(J^{-1}) \quad \text{for } m = 0, 1, 2, \dots$$

The Plemejl-Smithies formula (19) then implies (24).  $\square$

**Theorem 11.** *Using notation introduced in (5), suppose a real Jacobi matrix  $J$  obeys (15) and is invertible. Denote by  $\lambda_n(J)$ ,  $n \in \mathbb{N}$ , the eigenvalues of  $J$  (all of them are real and simple). Then  $L^{-1} - J^{-1} \in \mathcal{S}_1$ ,*

$$\sum_{n=1}^{\infty} \lambda_n(J)^{-2} < \infty, \quad (25)$$

and for the regularized characteristic function of  $J$  one has

$$H_J(z) = F_J(0) e^{bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n(J)}\right) e^{z/\lambda_n(J)} \quad (26)$$

where

$$b = \text{Tr}(L^{-1} - J^{-1}) = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_n(J)}\right).$$

*Proof.* Recall equation (23). Since  $L^{-1/2} \in \mathcal{J}_4$  and  $K \in \mathcal{J}_2$  one has, after some straightforward manipulations,

$$L^{-1} - J^{-1} = L^{-1/2}K(I + K)^{-1}L^{-1/2} \in \mathcal{J}_1. \quad (27)$$

By Lemma 10, the operator  $J^{-1}$  is Hermitian and Hilbert-Schmidt. This implies (25). Furthermore, by Lemma 9, formula (17) and Lemma 10,

$$\begin{aligned} H_J(z) &= \det_2(I + K - zL^{-1}) \\ &= \det_2(I + K) \det_2(I - z(I + K)^{-1}L^{-1}) \exp[z \operatorname{Tr}(K(I + K)^{-1}L^{-1})] \\ &= F_J(0) e^{bz} \det_2(I - zJ^{-1}). \end{aligned}$$

Here we have used (27) implying

$$\operatorname{Tr}(K(I + K)^{-1}L^{-1}) = \operatorname{Tr}(L^{-1/2}K(I + K)^{-1}L^{-1/2}) = \operatorname{Tr}(L^{-1} - J^{-1}) = b.$$

Finally, by formula (18),

$$\det_2(I - zJ^{-1}) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n(J)}\right) e^{z/\lambda_n(J)}.$$

This completes the proof.  $\square$

**Corollary 12.** *For each  $\epsilon > 0$  there is  $R_\epsilon > 0$  such that for  $|z| > R_\epsilon$ ,*

$$|H_J(z)| < \exp(\epsilon|z|^2). \quad (28)$$

*Proof.* Theorem 11, and particularly the product formula (26) implies that  $H_J(z)$  is an entire function of genus one. In that case the growth property (28) is known to be valid; see, for example, Theorem XI.2.6 in [2].  $\square$

*Example 13.* Put  $\lambda_n = n$  and  $w_n = w \neq 0$ ,  $\forall n \in \mathbb{N}$ . As shown in [11], the Bessel function of the first kind can be expressed as

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu + 1)} \mathfrak{F}\left(\left\{\frac{w}{\nu + k}\right\}_{k=1}^{\infty}\right), \quad (29)$$

as long as  $w, \nu \in \mathbb{C}$ ,  $\nu \notin -\mathbb{N}$ . Using (29) and that

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n},$$

where  $\gamma$  is the Euler constant, one gets  $H_J(z) = e^{\gamma z} w^z J_{-z}(2w)$ . Applying Theorem 11 to the Jacobi matrix in question one reveals the infinite product formula for the Bessel function considered as a function of its order. Assuming  $J_0(2w) \neq 0$ , the formula reads

$$\frac{w^z J_{-z}(2w)}{J_0(2w)} = e^{c(w)z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n(J)}\right) e^{z/\lambda_n(J)}$$

where

$$c(w) = \frac{1}{J_0(2w)} \sum_{k=0}^{\infty} (-1)^k \psi(k+1) \frac{w^{2k}}{(k!)^2},$$

$\psi(z) = \Gamma'(z)/\Gamma(z)$  is the digamma function, and the expression for  $c(w)$  is obtained by comparison of the coefficients at  $z$  on both sides.

## 4 Factorization in the coupling constant

Let  $x = \{x_n\}_{n=1}^\infty$  be a sequence of nonzero complex numbers belonging to the domain  $D$  defined in (1). Our goal in this section is to prove a factorization formula for the entire function

$$f(w) := \mathfrak{F}(wx), \quad w \in \mathbb{C}.$$

Let us remark that  $f(w)$  is even.

To this end, let us put  $v_k = \sqrt{x_k}$ ,  $\forall k$ , (any branch of the square root is suitable) and introduce the auxiliary Jacobi matrix

$$A = \begin{bmatrix} 0 & a_1 & 0 & 0 & \cdots \\ a_1 & 0 & a_2 & 0 & \cdots \\ 0 & a_2 & 0 & a_3 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \text{with } a_k = v_k v_{k+1}, \quad k \in \mathbb{N}. \quad (30)$$

Then  $A$  represents a Hilbert-Schmidt operator on  $\ell^2(\mathbb{N})$  with the Hilbert-Schmidt norm

$$\|A\|_2^2 = 2 \sum_{k=1}^{\infty} |a_k|^2 = 2 \sum_{k=1}^{\infty} |x_k x_{k+1}|.$$

The relevance of  $A$  to our problem comes from the equality

$$F_A(z) = \mathfrak{F}\left(\left\{\frac{x_k}{z}\right\}_{k=1}^{\infty}\right) = f(z^{-1}),$$

which can be verified with the aid of (4) and (9). Hence  $F_A(z)$  is analytic on  $\mathbb{C} \setminus \{0\}$ . By Theorem 2, the set of nonzero eigenvalues of  $A$  coincides with the zero set of  $F_A(z)$ . It even turns out that the algebraic multiplicity of a nonzero eigenvalue  $\zeta$  of  $A$  equals the multiplicity of  $\zeta$  as a root of the function  $F_A(z)$ , as one infers from the following proposition.

**Proposition 14.** *Under the same assumptions as in Theorem 2, the algebraic multiplicity of any nonzero eigenvalue  $\zeta$  of  $J$  is equal to the multiplicity of the root  $\zeta^{-1}$  of the entire function  $\varphi(z) = F_J(z^{-1}) = \mathfrak{F}(\{z\gamma_n^2\}_{n=1}^\infty)$ .*

*Proof.* Recall that  $\gamma_n \gamma_{n+1} = w_n$  and so, by the assumptions of Theorem 2,  $\{\gamma_n^2\} \in D$ . Denote again by  $P_N$ ,  $N \in \mathbb{N}$ , the orthogonal projection onto the subspace spanned by the first  $N$  vectors of the canonical basis in  $\ell^2(\mathbb{N})$ . From formula (7) we deduce that

$$\mathfrak{F}(\{z\gamma_n^2\}_{n=1}^N) = \det(I - zJ_N) = \det((I - zJ_N)e^{zJ_N}),$$

where  $J_N = P_N J P_N$ . Since  $P_N J P_N$  tends to  $J$  in the Hilbert-Schmidt norm, as  $N \rightarrow \infty$ , and by continuity of the generalized determinant as a functional on the space of Hilbert-Schmidt operators (see (21)) one immediately gets

$$\varphi(z) = \mathfrak{F}(\{z\gamma_n^2\}_{n=1}^\infty) = \det((I - zJ)e^{zJ}) = \det_2(I - zJ).$$

From (18) it follows that  $\varphi(z) = (1 - \zeta z)^m \tilde{\varphi}(z)$  where  $m$  is the algebraic multiplicity of  $\zeta$ ,  $\tilde{\varphi}(z)$  is an entire function and  $\tilde{\varphi}(\zeta^{-1}) \neq 0$ .  $\square$

The zero set of  $f(w)$  is at most countable and symmetric with respect to the origin. One can split  $\mathbb{C}$  into two half-planes so that the border line passes through the origin and contains no nonzero root of  $f$ . Fix one of the half-planes and enumerate all nonzero roots in it as  $\{\zeta_k\}_{k=1}^{N(f)}$ , with each root being repeated in the sequence according to its multiplicity. The number  $N(f)$  may be either a non-negative integer or infinity. Then

$$\operatorname{spec}_p(A) \setminus \{0\} = \{\pm \zeta_k^{-1}; k \in \mathbb{N}, k \leq N(f)\}.$$

Since  $A^2$  is a trace class operator one has, by Proposition 14 and Lidskii's theorem,

$$\sum_{k=1}^{N(f)} \frac{1}{\zeta_k^2} = \frac{1}{2} \operatorname{Tr} A^2 = \sum_{k=1}^{\infty} x_k x_{k+1}. \quad (31)$$

Moreover, the sum on the LHS converges absolutely, as it follows from Weyl's inequality [10, Thm. 1.15].

**Theorem 15.** *Let  $x = \{x_k\}_{k=1}^{\infty}$  be a sequence of nonzero complex numbers such that*

$$\sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty.$$

*Then zeros of the entire even function  $f(w) = \mathfrak{F}(wx)$  can be arranged into sequences*

$$\{\zeta_k\}_{k=1}^{N(f)} \cup \{-\zeta_k\}_{k=1}^{N(f)},$$

*with each zero being repeated according to its multiplicity, and*

$$f(w) = \prod_{k=1}^{N(f)} \left(1 - \frac{w^2}{\zeta_k^2}\right). \quad (32)$$

*Proof.* Equality (32) can be deduced from Hadamard's factorization theorem; see, for example, [2, Chp. XI]. In fact, the absolute convergence of the series  $\sum \zeta_k^{-2}$  in (31) means that the rank of  $f$  is at most 1. Furthermore, (2) implies that

$$|f(w)| \leq \exp\left(|w|^2 \sum_{k=1}^{\infty} |x_k x_{k+1}|\right),$$

and so the order of  $f$  is less than or equal to 2. Hadamard's factorization theorem tells us that the genus of  $f$  is at most 2. Taking into account that  $f$  is even and  $f(0) = 1$ , this means nothing but

$$f(w) = \exp(cw^2) \prod_{k=1}^{N(f)} \left(1 - \frac{w^2}{\zeta_k^2}\right)$$

for some  $c \in \mathbb{C}$ . Equating the coefficients at  $w^2$  one gets

$$-\sum_{k=1}^{\infty} x_k x_{k+1} = c - \sum_{k=1}^{N(f)} \frac{1}{\zeta_k^2}.$$

According to (31),  $c = 0$ . □

**Corollary 16.** *For any  $n \in \mathbb{N}$  (and recalling (10), (11)),*

$$\sum_{k=1}^{N(f)} \frac{1}{\zeta_k^{2n}} = n \sum_{m \in \mathcal{M}(n)} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} (x_{k+j-1} x_{k+j})^{m_j}. \quad (33)$$

*Proof.* Using Proposition 3, one can expand  $\log f(w)$  into a power series at  $w = 0$ . Applying log to (32) and equating the coefficients at  $w^{2n}$  gives (33). □

If the sequence  $\{x_k\}$  in Theorem 15 is positive one has some additional information about the zeros of  $f(w)$ . In that case the  $v_k$ s in (30) can be chosen positive, and so  $A$  is a self-adjoint Hilbert-Schmidt operator. The zero set of  $f$  is countable and all roots are real, simple and have no finite accumulation points. Enumerating positive zeros in ascending order as  $\zeta_k$ ,  $k \in \mathbb{N}$ , factorization (32) and identities (33) hold true. Since the first positive root  $\zeta_1$  is strictly smaller than all other positive roots, one has

$$\zeta_1 = \lim_{N \rightarrow \infty} \left( \sum_{m \in \mathcal{M}(N)} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} (x_{k+j-1} x_{k+j})^{m_j} \right)^{-1/(2N)}.$$

*Remark 17.* Still assuming the sequence  $\{x_k\}$  to be positive let  $g(z)$  be an entire function defined by

$$g(z) = 1 + \sum_{n=1}^{\infty} g_n z^n = \prod_{k=1}^{\infty} \left( 1 - \frac{z}{\zeta_k^2} \right),$$

i.e.  $g(w^2) = f(w)$ . In some particular cases the coefficients  $g_n$  may be known explicitly and then the spectral zeta function can be evaluated recursively. Put

$$\sigma(2n) = \sum_{k=1}^{\infty} \frac{1}{\zeta_k^{2n}}, \quad n \in \mathbb{N}.$$

Taking the logarithmic derivative of  $g(z)$  and equating coefficients at the same powers of  $z$  leads to the recurrence rule

$$\sigma(2) = -g_1, \quad \sigma(2n) = -ng_n - \sum_{k=1}^{n-1} g_{n-k} \sigma(2k) \quad \text{for } n > 1. \quad (34)$$

*Example 18.* Put  $x_k = (\nu + k)^{-1}$ , with  $\nu > -1$ . Recalling (29) and letting  $z = w/2$ , the factorization of the Bessel function [13],

$$\left(\frac{z}{2}\right)^{-\nu} \Gamma(\nu + 1) J_\nu(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,k}^2}\right),$$

is obtained as a particular case of Theorem 15. The zeros of  $J_\nu(z)$ , called  $j_{\nu,k}$ , also occur in the definition of the so called Rayleigh function [8],

$$\sigma_\nu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^s}, \quad \text{Re } s > 1.$$

Corollary 16 implies the formula

$$\sigma_\nu(2N) = 2^{-2N} N \sum_{k=1}^{\infty} \sum_{m \in \mathcal{M}(N)} \alpha(m) \prod_{j=1}^{d(m)} \left( \frac{1}{(j+k+\nu-1)(j+k+\nu)} \right)^{m_j}, \quad N \in \mathbb{N}.$$

*Example 19.* This examples is perhaps less commonly known and concerns the Ramanujan function, also interpreted as the  $q$ -Airy function by some authors [7, 14], and defined by

$$A_q(z) := {}_0\phi_1(; 0; q, -qz) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} (-z)^n, \quad (35)$$

where  ${}_0\phi_1(; b; q, z)$  is the basic hypergeometric series ( $q$ -hypergeometric series) and  $(a; q)_k$  is the  $q$ -Pochhammer symbol (see, for instance, [6]). In (35), we suppose that  $0 < q < 1$  and  $z \in \mathbb{C}$ . It has been shown in [11] that

$$A_q(w^2) = q \mathfrak{F} \left( \left\{ wq^{(2k-1)/4} \right\}_{k=1}^{\infty} \right).$$

Denote by  $0 < \zeta_1(q) < \zeta_2(q) < \zeta_3(q) < \dots$  the positive zeros of  $w \mapsto A_q(w^2)$  and put  $\iota_k(q) = \zeta_k(q)^2$ ,  $k \in \mathbb{N}$ . Then Theorem 15 tells us that the zeros of  $A_q(z)$  are exactly  $0 < \iota_1(q) < \iota_2(q) < \iota_3(q) < \dots$ , all of them are simple and

$$A_q(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{z}{\iota_k(q)} \right).$$

One has  $\{\iota_k(q)^{-1/2}; k \in \mathbb{N}\} = \text{spec}(\mathbf{A}(q)) \setminus \{0\}$  where  $\mathbf{A}(q)$  is the Hilbert-Schmidt operator in  $\ell^2(\mathbb{N})$  whose matrix is of the form (30), with  $a_k = q^{k/2}$ . Corollary 16 yields a formula for the spectral zeta function  $D_N(q)$  associated with  $A_q(z)$ , namely

$$D_N(q) := \sum_{k=1}^{\infty} \frac{1}{\iota_k(q)^N} = \frac{Nq^N}{1 - q^N} \sum_{m \in \mathcal{M}(N)} \alpha(m) q^{\epsilon_1(m)}, \quad N \in \mathbb{N},$$

where,  $\forall m \in \mathbb{N}^\ell$ ,  $\epsilon_1(m) = \sum_{j=1}^{\ell} (j-1) m_j$ . In accordance with (34), from the power series expansion of  $A_q(z)$  one derives the recurrence rule

$$D_n(q) = (-1)^{n+1} \frac{nq^{n^2}}{(q; q)_n} - \sum_{k=1}^{n-1} (-1)^k \frac{q^{k^2}}{(q; q)_k} D_{n-k}(q), \quad n = 1, 2, 3, \dots$$



Consider now a real Jacobi matrix  $J$  of the form (5) such that the diagonal sequence  $\{\lambda_n\}$  is semibounded. Suppose further that the off-diagonal elements  $w_n$  depend on a real parameter  $w$  as  $w_n = w\omega_n$ ,  $n \in \mathbb{N}$ , with  $\{\omega_n\}$  being a fixed sequence of positive numbers. Following physical terminology one may call  $w$  the coupling constant. Denote  $\lambda_{\inf} = \inf \lambda_n$ . Assume that

$$\sum_{n=1}^{\infty} \frac{\omega_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} < \infty$$

for some and hence any  $z < \lambda_{\inf}$ . For  $z < \lambda_{\inf}$ , Theorem 15 can be applied to the sequence

$$x_n(z) = \frac{\kappa_n^2}{\lambda_n - z}, \quad n \in \mathbb{N},$$

where  $\{\kappa_n\}$  is defined recursively by  $\kappa_1 = 1$ ,  $\kappa_n \kappa_{n+1} = \omega_n$ ; comparing to (6) one has  $\kappa_{2k-1} = \gamma_{2k-1}$ ,  $\kappa_{2k} = \gamma_{2k}/w$ . Let

$$F_J(z; w) = \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^{\infty}\right) = \mathfrak{F}(\{w x_n(z)\}_{n=1}^{\infty})$$

be the characteristic function of  $J = J(w)$ . We conclude that for every  $z < \lambda_{\inf}$  fixed, the equation  $F_J(z; w) = 0$  in the variable  $w$  has a countably many positive simple roots  $\zeta_k(z)$ ,  $k \in \mathbb{N}$ , enumerated in ascending order, and

$$F_J(z; w) = \prod_{k=1}^{\infty} \left(1 - \frac{w^2}{\zeta_k(z)^2}\right).$$

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